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AN OPERATOR-ANALYTIC APPROACH TO THE JACKSON NETWORK

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Abstract

Operator methods are used in this paper to systematically analyze the behavior of the Jackson network. Here, we consider rarely treated issues such as the transient behavior, and arbitrary subnetworks of the total system. By deriving the equations that govern an arbitrary subnetwork, we can see how the mean and variance for the queue length of one node as well as the covariance for two nodes vary in time.

We can estimate the transient behavior by deriving a stochastic upper bound for the joint distribution of the network in terms of a judicious choice of independent $M/M/1$ queue-length processes. The bound we derive is one that *cannot* be derived by a sample-path ordering of the two processes. Moreover, we can stochastically bound from below the process for the total number of customers in the network by an $M/M/1$ system also. These results allow us to approximate the network by the known transient distribution of the $M/M/1$ queue. The bounds are tight asymptotically for large-time behavior when every node exceeds heavy-traffic conditions.

STOCHASTIC ORDERING; TRANSIENT BEHAVIOR; TENSOR REPRESENTATION; SUBNETWORKS

1. Introduction

In this paper, we analyze the behavior of the Jackson network. Recall that this is an N-node network where the ith node is an $M/M/1$ queue with arrival and service rates λ_i and μ_i respectively. The queues are connected by an $N \times N$ switching matrix P where a customer, after receiving service at node i, leaves and joins the jth queue with probability p_{ij}. With probability $q_i = 1 - \Sigma_{j=1}^{N} p_{ij}$, however, it may decide to leave the network entirely. We shall always assume that $p_{ii} = 0$ for all i. Since the Jackson network is formally a collection of $M/M/1$ queues, we take liberties with Kendall notation and henceforth refer to an open, single-server, N-node Jackson network as $(M/M/1)^N$.

Let $Q_i(t)$ represent the random queue-length process for the ith node. Given the above formulation, the vector process $(Q_1(t), \cdots, Q_N(t))$ is Markov. In the spirit of the approach for the $M(t)/M(t)/1$ system (see Massey [5]), we apply

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operator methods to the Kolmogorov forwards equations for this Markov process. In Section 2, we use tensor product spaces to identify the primitive operators that generate the infinitesimal operator. This allows us to write down these equations in a compact but informative way. Section 3 demonstrates the power of the operator analytic approach by deriving new bounds for the transient behavior of the $(M/M/1)^N$ system in terms of the known transient behavior for the $M/M/1$ queue. We derive a stochastic bound of significance in that it is *not* equivalent to the usual stochastic ordering (Kamae, Krengel and O'Brien [3]). This means that this inequality cannot be derived by ordering sample paths.

Finally, in Section 4, we focus our attention on an arbitrary subnetwork of the system and consider the equations that govern it. We then use the subnetwork equations to derive differential equations for the mean and variance of the queue length associated with a node, as well as the covariance between any pair of nodes.

2. Tensor representation

Through the use of indicator functions, $Q_i(t)$ can be written as

$$Q_i(t) = \sum_{n=0}^{\infty} n \cdot I_{\{Q_i(t)=n\}}.$$

Now suppose that given the event $\{Q_i(t) = n\}$, we encode the corresponding information not as n, but as $\boldsymbol{e}_n$, the l_1-vector of all zeros except for a 1 in the nth place. We define this l_1-random variable as $\boldsymbol{q}_i(t, \omega)$ where

$$\boldsymbol{q}_i(t) = \sum_{n=0}^{\infty} \boldsymbol{e}_n \cdot I_{\{Q_i(t)=n\}}.$$

This representation is a way of encoding all of the possible events of $Q_i(t)$. If we take the expectation of $\boldsymbol{q}_i(t)$, then $E(\boldsymbol{q}_i(t))$ is a vector representation for the distribution of $Q_i(t)$.

We can do a similar representation for the entire network. Here, we let the quantity $\boldsymbol{p}(t)$ be an l_1-tensor of rank N with the $(n_1, \cdots, n_N)$th component being $p(n_1, \cdots, n_N; t)$ where

$$p(n_1, \cdots, n_N; t) = \Pr\{Q_1(t) = n_1, \cdots, Q_N(t) = n_N\} = E(I_{\{Q_1(t)=n_1\}} \cdots I_{\{Q_N(t)=n_N\}}).$$

Hence $\boldsymbol{p}(t)$ can be written compactly as

$$\boldsymbol{p}(t) = E(\boldsymbol{q}_1(t) \otimes \cdots \otimes \boldsymbol{q}_N(t)).$$

The Banach space that $\boldsymbol{p}(t)$ belongs to is $l_1^{(N)}$, or l_1 tensored with itself N times. If $\boldsymbol{g}_1, \cdots, \boldsymbol{g}_N$ are l_1 vectors, then $\boldsymbol{g}_1 \otimes \cdots \otimes \boldsymbol{g}_N = \bigotimes_{i=1}^{N} \boldsymbol{g}_i$ belongs to $l_1^{(N)}$. For the case $N = 2$, the tensor product has the following bilinear properties:

$$(g_1 + g_2)\otimes h = g_1 \otimes h + g_2 \otimes h$$
$$g \otimes (h_1 + h_2) = g \otimes h_1 + g \otimes h_2$$
$$(\alpha g)\otimes h = g \otimes (\alpha h) = \alpha (g \otimes h).$$

Since the e_n's form a basis for l_1, then tensor products of the form $e_{n_1} \otimes \cdots \otimes e_{n_N}$ constitute a basis for $l_1^{(N)}$. An element p of $l_1^{(N)}$ is *positive*, denoted $p \geqq 0$, if the coefficient for each basis element is non-negative. Similarly, for all p and q in $l_1^{(N)}$, we say that $p \geqq q$ if $p - q \geqq 0$. For all operators A and B on $l_1^{(N)}$, we say that $A \geqq 0$ if for all p in $l_1^{(N)}$, we have $pA \geqq 0$ whenever $p \geqq 0$, and $A \geqq B$ if $A - B \geqq 0$. We define an 'l_1-norm' on $l_1^{(N)}$ by summing the absolute values of the coefficients of each basis element. From this, it follows for example, that

$$\left| \bigotimes_{i=1}^{N} g_i \right|_1 = | g_1 \otimes \cdots \otimes g_N |_1 = \prod_{i=1}^{N} | g_i |_1.$$

For each $i = 1, \cdots, N$, let A_i be a bounded operator on l_1. We define $\otimes_{i=1}^N A_i$ so that

$$\left(\bigotimes_{i=1}^{N} g_i\right)\left[\bigotimes_{i=1}^{N} A_i\right] = \bigotimes_{i=1}^{N} g_i A_i.$$

Extending by linearity, $\otimes_{i=1}^N A_i$ becomes a bounded operator on $l_1^{(N)}$. We now have the means to write down succintly the birth and death equations for $(M/M/1)^N$.

Proposition 2.1. *Let the components of $p(t)$ be the joint distribution for the queue lengths of $(M/M/1)^N$. Then*

$$\frac{d}{dt} p(t) = p(t)A. \tag{2.1}$$

Using R and L, the right- and left-shift operators on l_1, we can write A as

$$A = \sum_{i=1}^{N} \left[\lambda_i R_i + \mu_i q_i L_i + \sum_{j=1}^{N} \mu_i p_{ij} L_i R_j - \lambda_i I - \mu_i L_i R_i \right] \tag{2.2}$$

where

$$R_i = I \otimes \cdots \otimes R \otimes \cdots \otimes I \qquad (i\text{th place})$$
$$L_i = I \otimes \cdots \otimes L \otimes \cdots \otimes I \qquad (i\text{th place}).$$

Proof. This representation can be easily verified by noting that these birth and death equations are derived from thinking of flows in and out of states. For example, observe that R_i acts on $p(t)$ through the right-shift operator acting on $q_i(t)$ and leaving the other $q_j(t)$'s fixed. So if a customer arrives with rate λ_i to the ith node at state

$$\{Q_1(t) = n_1, \cdots, Q_i(t) = n_i - 1, \cdots, Q_N(t) = n_N\}$$

then we have made the transition to state

$$\{Q_1(t) = n_1, \cdots, Q_i(t) = n_i, \cdots, Q_N(t) = n_N\}.$$

Furthermore, the properties of the right-shift operator take care of the technicalities that arise when $n_i = 0$.

Each term in $\boldsymbol{A}$ has a similar interpretation and by inspection we have properly encoded all of the birth and death equations.

Using this notation makes it easier to analyze what would otherwise be an unwieldy set of equations. We shall demonstrate the benefits of this approach in the following sections.

3. Stochastic bounds for the transient behavior

For any given $(\boldsymbol{M}/\boldsymbol{M}/\boldsymbol{1})^N$ system, we can derive the following set of stochastic bounds.

Theorem 3.1. *Let $X_1(t), \cdots, X_N(t)$ be a collection of independent* $\boldsymbol{M}/\boldsymbol{M}/\boldsymbol{1}$ *queue-length processes where $X_i(t)$ has arrival rate $\lambda_i + \sum_{j=1}^N \mu_j p_{ji}$, service rate μ_i and $X_i(0) = Q_i(0)$, then*

$$\Pr\{Q_1(t) \geqq n_1, \cdots, Q_N(t) \geqq n_N\} \leqq \prod_{i=1}^{N} \Pr\{X_i(t) \geqq n_i\} \tag{3.1}$$

for all $t > 0$, and all non-negative integers $n_1, \cdots, n_N$.

Now let $Y(t)$ be an $\boldsymbol{M}/\boldsymbol{M}/\boldsymbol{1}$ *queue-length process with arrival rate $\sum_{i=1}^N \lambda_i$, service rate $\sum_{i=1}^N \mu_i q_i$, and $Y(0) = \sum_{i=1}^N Q_i(0)$, then*

$$\Pr\left\{\sum_{i=1}^{N} Q_i(t) \geqq n\right\} \geqq \Pr\{Y(t) \geqq n\}$$

for all t and all non-negative integers n.

If we use the notion of stochastic ordering in the sense of Kamae, Krengel and O'Brien [3] (which we will denote by $\leqq_{st}$), then Theorem 3.1 says that $Y(t) \leqq_{st} \sum_{i=1}^N Q_i(t)$. Despite the suggestive inequality (3.1) between the joint Markov processes $(Q_1(t), \cdots, Q_N(t))$ and $(X_1(t), \cdots, X_N(t))$, however, it is *never* the case for non-trivial $(\boldsymbol{M}/\boldsymbol{M}/\boldsymbol{1})^N$ networks ($p_{ij} \neq 0$ for some i and j) that $(Q_1(t), \cdots, Q_N(t)) \leqq_{st} (X_1(t), \cdots, X_N(t))$.

Theorem 3.2. *For any $(\boldsymbol{M}/\boldsymbol{M}/\boldsymbol{1})^N$ system with $(Q_1(t), \cdots, Q_N(t))$ and $(X_1(t), \cdots, X_N(t))$ defined as in Theorem* 3.1, *the result $(Q_1(t), \cdots, Q_N(t)) \not\leqq_{st} (X_1(t), \cdots, X_N(t))$ holds if and only if $p_{ij} \neq 0$ for some i and j.*

Notice that Theorem 3.2 is equivalent to saying that $(Q_1(t), \cdots, Q_N(t))$ and $(X_1(t), \cdots, X_N(t))$ can be stochastically ordered using $\leqq_{st}$ if and only if they are identical in distribution. By the equivalence results of Kamae, Krengel and O'Brien [3], Theorem 3.2 also says that (3.1) will *never* yield a strict sample path ordering between the two processes. In [6], similar bounds are derived for an open network with different classes of customer.

Proof of Theorem 3.2. The state space for the two processes is Z_+^N, the set of non-negative integer N-tuples. The stochastic ordering in question is induced by the vector ordering that we defined previously. Suppose that $p_{ij} \neq 0$ for some i and j, then $q_i \neq 1$ for some i. Given $\boldsymbol{n} = (n_1, \cdots, n_N)$ in Z_+^N with $n_i \neq 0$, define the following subset of Z_+^N:

$$\Gamma_i = \{\boldsymbol{m} \mid \boldsymbol{m} \in Z_+^N \text{ with } \boldsymbol{m} \geqq \boldsymbol{n} - \boldsymbol{e}_j \text{ or } \boldsymbol{m} \geqq \boldsymbol{n} - \boldsymbol{e}_i + \boldsymbol{e}_j \text{ for some } j \neq i\}$$

where the $\boldsymbol{e}_i$'s are the basis unit vectors of Z_+^N.

Now set $Q_i(0) = X_i(0) = n_i$ for all i. Γ_i is an *increasing set*, that is if $\boldsymbol{m} \in \Gamma_i$ and $\boldsymbol{m}' \geqq \boldsymbol{m}$, then $\boldsymbol{m}' \in \Gamma_i$. So if a relation like $\leqq_{st}$ holds between the two processes, we should have for all $t \geqq 0$

$$\Pr\{(Q_1(t), \cdots, Q_N(t)) \in \Gamma_i\} \leqq \Pr\{(X_1(t), \cdots, X_N(t)) \in \Gamma_i\}.$$

Both processes are Markov, so if we subtract the number one from both sides, divide by t, and let t go to 0, we get the respective infinitesimal rates of flow out of the set Γ_i from state $\boldsymbol{n}$. However, this gives us $-\mu_i q_i \leqq -\mu_i$. Since q_i is strictly less than 1, we get a contradiction. Therefore the relation $\leqq_{st}$ cannot hold.

On the other hand, if $p_{ij} = 0$ for all i and j, then $(Q_1(t), \cdots, Q_N(t))$ and $(X_1(t), \cdots, X_N(t))$ have the same distribution so the relation $\leqq_{st}$ holds trivially.

In a stochastic sense, Theorem 3.1 states that $(X_1(t), \cdots, X_N(t))$ is an 'upper bound' for $(Q_1(t), \cdots, Q_N(t))$ and $Y(t)$ is a 'lower bound' for $\Sigma_{i=1}^N Q_i(t)$. Moreover, these bounds hold for all time so we have estimates for the *transient* behavior of an $(\boldsymbol{M}/\boldsymbol{M}/\mathbf{1})^N$ system in terms of the transient distribution for an $\boldsymbol{M}/\boldsymbol{M}/\mathbf{1}$ queue which is known (see Gross and Harris [1]). The proof of this theorem will illustrate the applicability of operator methods to queueing networks.

This first step is to create two additional operators. If $\boldsymbol{L}$ is the left-shift operator on l_1, then $(\boldsymbol{I} - \boldsymbol{L})^{-1}$ is a positive unbounded operator on l_1 where

$$\boldsymbol{e}_n (\boldsymbol{I} - \boldsymbol{L})^{-1} = \boldsymbol{e}_n \cdot \sum_{k=0}^{\infty} \boldsymbol{L}^k.$$

This operator is well defined on the basis elements, so $(\boldsymbol{I} - \boldsymbol{L})^{-1}$ has a dense domain. Now define $\boldsymbol{K}_i$ on $l_1^{(N)}$ where

$$\boldsymbol{K}_i = \boldsymbol{I} \otimes \cdots \otimes (\boldsymbol{I} - \boldsymbol{L})^{-1} \otimes \cdots \otimes \boldsymbol{I} \qquad (i\text{th place}),$$

let $\boldsymbol{K}_{(i)} = \Pi_{j \neq i} \boldsymbol{K}_j$, and define $\boldsymbol{K} = \Pi_{i=1}^N \boldsymbol{K}_i$. Now let $\boldsymbol{S}$ be an operator that maps $l_1^{(N)}$ to l_1 as follows:

$$\left(\bigotimes_{i=1}^N \boldsymbol{e}_{n_i} \right) \boldsymbol{S} = \boldsymbol{e}_m$$

where $m = \Sigma_{i=1}^N n_i$. Unlike $\boldsymbol{K}$, $\boldsymbol{S}$ is a bounded operator but both of them are positive.

Proposition 3.3. *$\boldsymbol{K}$ and $\boldsymbol{S}$ have the following properties*:

(1) $\boldsymbol{R}_i \boldsymbol{K} = \boldsymbol{K} + \boldsymbol{K}_{(i)} \boldsymbol{R}_i$

(2) $\boldsymbol{L}_i \boldsymbol{K} = \boldsymbol{K} - \boldsymbol{K}_{(i)}$

(3) $\boldsymbol{R}_i \boldsymbol{S} = \boldsymbol{S} \boldsymbol{R}$

(4) $\boldsymbol{L}_i \boldsymbol{S} \leqq \boldsymbol{S} \boldsymbol{L}$.

Proof. By definition, we can think of $\boldsymbol{K}_i$ as $\boldsymbol{I} + \boldsymbol{L}_i + \boldsymbol{L}_i^2 + \cdots$, therefore we have $\boldsymbol{L}_i \boldsymbol{K}_i = \boldsymbol{K}_i - \boldsymbol{I}$ and $\boldsymbol{R}_i \boldsymbol{K}_i = \boldsymbol{K}_i + \boldsymbol{R}_i$ since $\boldsymbol{R}_i \boldsymbol{L}_i = \boldsymbol{I}$. The rest follows from the definition of $\boldsymbol{K}$.

To prove the identities involving $\boldsymbol{S}$, one need only show it for the basis elements and the rest will follow by linearity. For example, if $m = \Sigma_{i=1}^N n_i$, then

$$\left(\bigotimes_{i=1}^N \boldsymbol{e}_{n_i} \right) \boldsymbol{R}_j \boldsymbol{S} = \boldsymbol{e}_{m+1} = \boldsymbol{e}_m \boldsymbol{R} = \left(\bigotimes_{i=1}^N \boldsymbol{e}_{n_i} \right) \boldsymbol{S} \boldsymbol{R} ;$$

by a similar argument we prove (4).

The purpose for $\boldsymbol{K}$ and $\boldsymbol{S}$ will now be made clear.

Proposition 3.4. *If $\boldsymbol{p}(t)$ is the rank-N tensor belonging to $l_1^{(N)}$ that represents the joint distribution for an $(\boldsymbol{M}/\boldsymbol{M}/1)^N$ system, then*

$$\boldsymbol{p}(t)\boldsymbol{K} = \sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \Pr\{Q_1(t) \geqq n_1, \cdots, Q_N(t) \geqq n_N\} \bigotimes_{i=1}^N \boldsymbol{e}_{n_i}$$

$$\boldsymbol{p}(t)\boldsymbol{S} = \sum_{n=0}^{\infty} \Pr\{Q_1(t) + \cdots + Q_N(t) = n\} \boldsymbol{e}_n .$$

Proof. Recall the definition of $\boldsymbol{q}_i(t)$ and apply $(\boldsymbol{I} - \boldsymbol{L})^{-1}$ to it

$$\begin{aligned} \boldsymbol{q}_i(t)(\boldsymbol{I} - \boldsymbol{L})^{-1} &= \sum_{j=0}^{\infty} I_{\{Q_i(t)=j\}} \boldsymbol{e}_j \cdot \sum_{k=0}^{\infty} \boldsymbol{L}^k \\ &= \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} I_{\{Q_i(t)=j\}} \boldsymbol{e}_{j-k} \\ &= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} I_{\{Q_i(t)=m+k\}} \boldsymbol{e}_m \\ &= \sum_{m=0}^{\infty} I_{\{Q_i(t) \geqq m\}} \boldsymbol{e}_m . \end{aligned}$$

It then follows that

$$\left(\bigotimes_{i=1}^{N} q_i(t)\right) K = \bigotimes_{i=1}^{N} q_i(t)(I-L)^{-1}$$
$$= \sum_{m_1=0}^{\infty} \cdots \sum_{m_N=0}^{\infty} I_{\{Q_1(t)\geqq m_1\}} \cdots I_{\{Q_N(t)\geqq m_N\}} \bigotimes_{i=1}^{N} e_{m_i}.$$

The first identity then follows by taking expectations. The second equation follows similarly after applying S to $\otimes_{i=1}^{N} q_i(t)$.

A key element in proving Theorem 3.1 is to compare two semigroups. We now derive an identity essential towards achieving that end.

Lemma 3.5. *Let* A *and* B *be bounded operators on some Banach space, then*

$$\exp(tA) - \exp(tB) = \int_0^t \exp(sA)(A-B)\exp((t-s)B)ds.$$

Proof. Let $\Omega(t) = \exp(tA) - \exp(tB)$. This operator is differentiable with respect to t so

$$\frac{d}{dt}\Omega(t) = \exp(tA)A - \exp(tB)B$$
$$= \Omega(t)B + \exp(tA)(A-B).$$

Now $\Omega(0) = 0$, so by Duhamel's principle, we have the desired results.

Now we can prove the main theorem.

Proof of Theorem 3.1. Let A be the generator given for the $(M/M/1)^N$ system as in (2.2). Now set $\lambda'_i = \lambda_i + \Sigma_{j=1}^{N} \mu_j p_{ji}$ and make $A_i = \lambda'_i R_i + \mu_i L_i - \lambda'_i I - \mu_i L_i R_i$. A collection of independent $M/M/1$ queues, namely $(X_1(t), \cdots, X_N(t))$, is a trivial case of an $(M/M/1)^N$ system with all of the p_{ij}'s equal to 0. Therefore $\Sigma_{i=1}^{N} A_i$ is the generator for the joint process of $X_i(t)$'s.

By (2.1), $p(t)$ for $(M/M/1)^N$ equals $p(0)\exp(tA)$. By Proposition 3.4, we want to show that

$$p(0)\exp(tA)K \leqq p(0)\exp\left(t \cdot \sum_{i=1}^{N} A_i\right) K. \tag{3.2}$$

Since $p(0)$ is always a positive vector, we need only show that

$$\exp(tA)K \leqq \exp\left(t \cdot \sum_{i=1}^{N} A_i\right) K.$$

Keilson proves in [4] that a semigroup like $\exp(tA)$ yields a positive operator for every instance of t if and only if every non-diagonal element of the generator, namely A, is positive. From this is is clear that $\exp(tA)$ and $\exp(t\cdot\Sigma_{i=1}^N A_i)$ are positive. Moreover, by Proposition 3.3, we have that

$$K^{-1}A_iK=\lambda_i'R_i+\mu_iL_i^2R_i-(\lambda_i'+\mu_i)L_iR_i$$

and so $K^{-1}\exp(t\cdot\Sigma_{i=1}^N A_i)K$ is positive also. By Lemma 3.5, we have

$$\exp(tA)K-\exp\left(t\cdot\sum_{i=1}^N A_i\right)K$$
$$=\int_0^t \exp(sA)\left(A-\sum_{i=1}^N A_i\right)K\cdot K^{-1}\exp\left((t-s)\sum_{i=1}^N A_i\right)Kds$$

so to prove (3.2), it is sufficient to show that $\Sigma_{i=1}^N A_iK\geqq AK$.

$$\begin{aligned} AK&=\sum_{i=1}^N\left[\lambda_iR_i+\mu_iq_iL_i+\sum_{j=1}^N\mu_ip_{ij}L_iR_j-\lambda_iI-\mu_iL_iR_i\right]K\\ &=\sum_{i=1}^N\left[\lambda_iK_{(i)}R_i+\mu_iq_iL_iK+\sum_{j=1}^N\mu_ip_{ij}L_i(K_{(j)}R_j+K)-\mu_iL_i(K_{(i)}R_j+K)\right]\\ &=\sum_{i=1}^N\left[\lambda_iK_{(i)}R_i+\sum_{j=1}^N\mu_ip_{ij}L_iK_{(j)}R_j-\mu_iK_{(i)}L_iR_i\right]\\ &=\sum_{i=1}^N\left[\left(\lambda_i+\sum_{j=1}^N\mu_jp_{ji}\right)K_{(i)}R_i-\mu_iK_{(i)}L_iR_i-\sum_{j=1}^N\mu_ip_{ij}K_{(i,j)}L_iR_j\right]\end{aligned}\tag{3.3}$$

where $K_{(i,j)}=\Pi_{m\neq i,j}K_m$. But $K_{(i,j)}L_iR_j$ is a positive operator, so

$$\begin{aligned} AK&\leqq\sum_{i=1}^N[\lambda_i'K_{(i)}R_i-\mu_iK_{(i)}L_iR_i]\\ &\leqq\sum_{i=1}^N[\lambda_i'R_i(I-L_i)-\mu_iL_iR_i(I-L_i)]K\\ &\leqq\sum_{i=1}^N A_iK.\end{aligned}$$

To prove the second part of the theorem, let $\lambda^*=\Sigma_{i=1}^N\lambda_i$, $\mu^*=\Sigma_{i=1}^N\mu_iq_i$ and $B=\lambda^*R+\mu^*L-\lambda^*I-\mu^*LR$. B is an operator on l_1 and is the generator for $Y(t)$. We want to show that

$$\exp(tA)SK\geqq S\exp(tB)K.$$

We can modify the proof of Lemma 3.5 to say that

$$\exp(tA)S-S\exp(tB)=\int_0^t\exp(sA)(AS-SB)\exp((t-s)B)ds.$$

Again, $K^{-1}\exp(tB)K\geqq 0$ so we need only show that $ASK\geqq SBK$. Note here that $K=(I-L)^{-1}$.

$$\begin{aligned}
\boldsymbol{ASK} &= \sum_{i=1}^{N} \left[\lambda_i \boldsymbol{R}_i + \mu_i q_i \boldsymbol{L}_i + \sum_{j=1}^{N} \mu_i p_{ij} \boldsymbol{L}_i \boldsymbol{R}_j - \lambda_i \boldsymbol{I} - \mu_i \boldsymbol{L}_i \boldsymbol{R}_i \right] \boldsymbol{SK} \\
&= \sum_{i=1}^{N} \left[\lambda_i \boldsymbol{SR} + \mu_i q_i \boldsymbol{L}_i \boldsymbol{S} + \sum_{j=1}^{N} \mu_i p_{ij} \boldsymbol{L}_i \boldsymbol{SR} - \lambda_i \boldsymbol{S} - \mu_i \boldsymbol{L}_i \boldsymbol{SR} \right] \boldsymbol{K} \\
&= \sum_{i=1}^{N} [\lambda_i \boldsymbol{S}(\boldsymbol{R} - \boldsymbol{I}) - \mu_i q_i \boldsymbol{L}_i \boldsymbol{S}(\boldsymbol{R} - \boldsymbol{I})] \boldsymbol{K} \\
&= \left(\sum_{i=1}^{N} \lambda_i \right) \boldsymbol{SR} - \sum_{i=1}^{N} \mu_i q_i \boldsymbol{L}_i \boldsymbol{SR} \\
&\geqq \boldsymbol{S}(\lambda^* \boldsymbol{R} - \mu^* \boldsymbol{LR}) \\
&\geqq \boldsymbol{SBK}
\end{aligned}$$

and this proves the theorem.

We now show when these bounds are tight.

Theorem 3.6. Suppose $\lambda_i + \Sigma_{j=1}^N \mu_j p_{ji} > \mu_i$ for all i, then as $t \to \infty$

$$\lim_{t\to\infty} \frac{1}{t} E(Q_i(t)) = \lambda_i + \sum_{j=1}^{N} \mu_j p_{ji} - \mu_i.$$

Proof. By Theorem 3.1, $E(Q_i(t)) \leqq E(X_i(t))$ and $\Sigma_{i=1}^N E(Q_i(t)) = E(\Sigma_{i=1}^N Q_i(t)) \geqq E(Y_i(t))$. But the $X_i(t)$'s and $Y(t)$ are $\boldsymbol{M/M/1}$ queue-length processes, and their asymptotic behavior is known (see Massey [5]). By the hypothesis, we have as $t \to \infty$

$$E(X(t)) = \left(\lambda_i + \sum_{j=1}^{N} \mu_j p_{ji} - \mu_i \right) t + O(1). \tag{3.4}$$

Notice that

$$\sum_{i=1}^{N} \left(\lambda_i + \sum_{j=1}^{N} \mu_j p_{ji} - \mu_i \right) = \sum_{i=1}^{N} [\lambda_i + \mu_i(1 - q_i) - \mu_i] = \sum_{i=1}^{N} \lambda_i - \sum_{i=1}^{N} \mu_i q_i$$

so the hypothesis also implies that $\Sigma_{i=1}^N \lambda_i > \sum_{i=1}^{N} \mu_i q_i$, hence

$$E(Y(t)) = \left(\sum_{i=1}^{N} \lambda_i - \sum_{i=1}^{N} \mu_i q_i \right) t + O(1). \tag{3.5}$$

By (3.4) we have

$$\limsup_{t\to\infty} \frac{1}{t} E(Q_i(t)) \leqq \lambda_i + \sum_{j=1}^{N} \mu_j p_{ji} - \mu_i. \tag{3.6}$$

On the other hand, by (3.5) we get

$$\liminf_{t\to\infty} \frac{1}{t} \sum_{i=1}^{N} E(Q_i(t)) \geqq \sum_{i=1}^{N} \lambda_i - \sum_{i=1}^{N} \mu_i q_i \geqq \sum_{i=1}^{N} \left(\lambda_i + \sum_{j=1}^{N} \mu_j p_{ji} - \mu_i \right)$$

but (3.6) implies that

$$\limsup_{t\to\infty} \frac{1}{t} \sum_{i=1}^{N} E(Q_i(t)) \leqq \sum_{i=1}^{N} \left(\lambda_i + \sum_{j=1}^{N} \mu_j p_{ji} - \mu_i\right).$$

Therefore,

$$\lim_{t\to\infty} \frac{1}{t} \sum_{i=1}^{N} E(Q_i(t)) = \sum_{i=1}^{N} \lambda_i - \sum_{i=1}^{N} \mu_i q_i. \tag{3.7}$$

From this it follows that

$$\sum_{i=1}^{N} \limsup_{t\to\infty} \frac{1}{t} E(Q_i(t)) \geqq \sum_{i=1}^{N} \left(\lambda_i + \sum_{j=1}^{N} \mu_j p_{ji} - \mu_i\right).$$

Since (3.6) holds, we must have

$$\limsup_{t\to\infty} \frac{1}{t} E(Q_i(t)) = \lambda_i + \sum_{j=1}^{N} \mu_j p_{ji} - \mu_i. \tag{3.8}$$

Finally, (3.7) combined with (3.8) implies that

$$\lim_{t\to\infty} \frac{1}{t} E(Q_i(t)) = \lambda_i + \sum_{j=1}^{N} \mu_j p_{ji} - \mu_i \quad \text{for all } i.$$

4. Subnetworks and moment formulas

Define $\mathbf{1}_I$ as a linear map from $l_1^{(N)}$ to $\bigotimes_{i\notin I} l_1$ where

$$\left(\bigotimes_{i=1}^{N} g_i\right) \cdot \mathbf{1}_I = \prod_{i\in I} (g_i \cdot \mathbf{1}) \cdot \bigotimes_{j\notin I} g_j$$

we then extend to all of $l_1^{(N)}$ by linearity. We shall be doing calculations with $\mathbf{1}_I$, $\boldsymbol{R}_i$, and $\boldsymbol{L}_i$. Here, we show for example, how $\mathbf{1}_I$ and $\boldsymbol{R}_i$ interact.

$$\boldsymbol{R}_i \mathbf{1}_I = \begin{cases} \mathbf{1}_I. & \text{if } i \in I; \\ \mathbf{1}_I \boldsymbol{R}_i. & \text{if } i \notin I. \end{cases}$$

Technically, in the second case, the $\boldsymbol{R}_i$ on the left is an operator on $l_1^{(N)}$, whereas the one on the right acts on $\bigotimes_{i\notin I} l_1$. We shall abuse notation and let $\boldsymbol{R}_i$ denote both.

Proposition 4.1. *For any $(M/M/1)^N$ system, let I be any subset of the index set $\{1, 2, \cdots, N\}$ and define $p^I(t)$ as*

$$p^I(t) = E\left(\bigotimes_{i\in I} q_i(t)\right).$$

Then $p^I(t)$ represents the joint distribution for the subnetwork of $(M/M/1)^N$ indexed by I and it satisfies the equation

$$\frac{d}{dt}\, p^I(t) = p^I(t)A_I + \sum_{i\in I}\sum_{j\notin I} \mu_j p_j^I(0; t) p_{ji}(I - R_i)$$

where $p_j^I(0; t)$ represents the joint distribution of $\{Q_i(t)\}_{i\in I}$ and the event $\{Q_j(t) = 0\}$, and

$$A_I = \sum_{i\in I}\Big[\Big(\lambda_i + \sum_{j\notin I}\mu_j p_{ji}\Big) R_i + \mu_i\Big(q_i + \sum_{j\notin I} p_{ij}\Big) L_i$$
$$+ \sum_{j\in I}\mu_i p_{ij} L_i R_j - \mu_i L_i R_i - \Big(\lambda_i + \sum_{j\notin I}\mu_j p_{ji}\Big) I\Big].$$

Proof. Notice that $|q_i(t)|_1 = q_i(t)\cdot \mathbf{1} = 1$. Since $p^I(t) = E(\otimes_{i\in I} q_i(t))$, we define (I) to be the complement of the index set I, and then $p^{I}(t) \doteq p(t)\cdot \mathbf{1}_{(I)}$. We now apply $\mathbf{1}_{(I)}$ to (2.1) making use of the following identity:

$$R_i \mathbf{1}_{(I)} = \begin{cases} \mathbf{1}_{(I)} R_i. & \text{if } i \in I; \\ \mathbf{1}_{(I)}. & \text{if } i \notin I \end{cases}$$

where $L_i \mathbf{1}_{(I)} = \mathbf{1}_{(I)} L_i$ abusing notation, if i belongs to I, otherwise, we leave it as it is.

$$A\,\mathbf{1}_{(I)} = \mathbf{1}_{(I)}\sum_{i\in I}\lambda_i R_i + \mathbf{1}_{(I)}\sum_{i\notin I}\lambda_i + \mathbf{1}_{(I)}\sum_{i\in I}\mu_i q_i L_i + \sum_{i\notin I}\mu_i q_i L_i \mathbf{1}_{(I)}$$
$$+ \sum_{i=1}^{N}\mu_i L_i \mathbf{1}_{(I)}\Big(\sum_{j\in I} p_{ij} R_j + \sum_{j\notin I} p_{ij}\Big) - \mathbf{1}_{(I)}\sum_{i\in I}\mu_i L_i R_i - \sum_{i\notin I}\mu_i L_i \mathbf{1}_{(I)} - \mathbf{1}_{(I)}\sum_{i=1}^{N}\lambda_i I$$
$$= \mathbf{1}_{(I)}\sum_{i\in I}\Big[\lambda_i R_i + \mu_i q_i L_i - \mu_i L_i R_i + \mu_i L_i\Big(\sum_{j\in I} p_{ij} R_j + \sum_{j\notin I} p_{ij}\Big) - \lambda_i I\Big]$$
$$+ \sum_{i\notin I}\mu_i L_i \mathbf{1}_{(I)}\Big(\sum_{j\in I} p_{ij} R_j + \sum_{j\notin I} p_{ij}\Big) + \sum_{i\notin I}\mu_i (q_i - 1) L_i \mathbf{1}_{(I)}$$
$$= \mathbf{1}_{(I)}\Big[\sum_{i\in I}\lambda_i R_i + \mu_i\Big(q_i + \sum_{j\notin I} p_{ij}\Big) L_i - \mu_i L_i R_i + \sum_{j\in I}\mu_i p_{ij} L_i R_j - \lambda_i I\Big]$$
$$- \sum_{i\in I}\sum_{j\notin I}\mu_j L_j \mathbf{1}_{(I)} p_{ji}(I - R_i).$$

Now we observe that $p_j^I(0; t) = p(t)(I - L_j)\mathbf{1}_{(I)}$ for j not in I, and the rest follows.

Proposition 4.2. *For any $(M/M/1)^N$ system, the following differential equations hold for the mean and variance of the i-th node, as well as the covariance of the i-th and j-th nodes:*

$$\frac{d}{dt}E(Q_i(t)) = \lambda_i + \sum_{j\neq i}^{N} \mu_j p_{ji} - \mu_i + \mu_i p_i(0;t) - \sum_{j\neq i}^{N} \mu_j p_j(0;t) p_{ji}$$

$$\frac{d}{dt}\mathrm{Var}(Q_i(t)) = \lambda_i + \sum_{j\neq i}^{N} \mu_j p_{ji} + \mu_i - \mu_i(2\Gamma_{i;i}(t) + p_i(0;t))$$

$$+ \sum_{j\neq i}^{N} \mu_i p_{ji}(2\Gamma_{i;j}(t) - p_j(0;t))$$

$$\frac{d}{dt}\mathrm{Cov}(Q_i(t), Q_j(t)) = -(\mu_i p_{ij} + \mu_j p_{ji}) + \mu_i p_{ij}(\Gamma_{i;i}(t) + p_i(0;t))$$

$$+ \mu_j p_{ji}(\Gamma_{j;j}(t) + p_j(0;t)) - \mu_i \Gamma_{j;i}(t) - \mu_j \Gamma_{i;j}(t)$$

$$+ \sum_{k\neq i,j}^{N} \mu_k p_{ki} \Gamma_{i;k}(t) + \sum_{k\neq i,j}^{N} \mu_k p_{kj} \Gamma_{j;k}(t)$$

where $p_i(0;t) = \Pr\{Q_i(t) = 0\}$, $\Gamma_{i;j}(t) = E(Q_i(t))p_j(0;t) - E(Q_i(t); Q_j(t) = 0)$, *and in particular,* $\Gamma_{i;i}(t) = E(Q_i(t))p_i(0;t)$.

Proof. For each $i = 1, \cdots, N$; let $\boldsymbol{p}_i(t) = \boldsymbol{p}^{\{i\}}(t)$. By Proposition 4.1, we have

$$\frac{d}{dt}\boldsymbol{p}_i(t) = \boldsymbol{p}_i(t)\boldsymbol{A}_i + \sum_{j\neq i}^{N} \mu_j \boldsymbol{p}_j^i(0;t) p_{ji}(\boldsymbol{I} - \boldsymbol{R}) \tag{4.1}$$

where

$$\boldsymbol{A}_i = \left(\lambda_i + \sum_{j\neq i}^{N} \mu_j p_{ji}\right)\boldsymbol{R} + \mu_i \boldsymbol{L} - \mu_i \boldsymbol{L}\boldsymbol{R} - \left(\lambda_i + \sum_{j\neq i}^{N} \mu_j p_{ji}\right)\boldsymbol{I}$$

since $p_{ii} = 0$. We shall identify the necessary calculations for each formula.

Case 1: $E(Q_i(t))$. Let $\boldsymbol{n} = [0,1,2,\cdots]^{\mathrm{T}}$ then $\boldsymbol{p}_i(t)\cdot\boldsymbol{n} = E(Q_i(t))$. By inspection, we can see that

$$\boldsymbol{A}_i\boldsymbol{n} = \left(\lambda_i + \sum_{j\neq i}^{N} \mu_j p_{ji} - \mu_i\right)\boldsymbol{1} + \mu_i \boldsymbol{e}_0$$

$$(\boldsymbol{I} - \boldsymbol{R})\boldsymbol{n} = -\boldsymbol{1}.$$

Now apply $\boldsymbol{n}$ to (4.1). Since $\boldsymbol{p}_i(t)\cdot\boldsymbol{e}_0 = p_i(0;t)$ and $\boldsymbol{p}_j^i(0;t)\cdot\boldsymbol{1} = p_j(0;t)$, we have the formula.

Case 2: $\mathrm{Var}(Q_i(t))$. Let $\boldsymbol{n}^2 = [0^2, 1^2, 2^2, \cdots]^{\mathrm{T}}$, then $\boldsymbol{p}_i(t)\cdot\boldsymbol{n}^2 = E(Q_i(t)^2)$. Furthermore,

$$\boldsymbol{A}_i\boldsymbol{n}^2 = 2\left(\lambda_i + \sum_{j\neq i}^{N} \mu_j p_{ji} - \mu_i\right)\boldsymbol{n} + \left(\lambda_i + \sum_{j\neq i}^{N} \mu_j p_{ji} + \mu_i\right)\boldsymbol{1} - \mu_i \boldsymbol{e}_0$$

$$(\boldsymbol{I} - \boldsymbol{R})\boldsymbol{n}^2 = -(\boldsymbol{1} + 2\boldsymbol{n}).$$

Since $\boldsymbol{p}_j^i(0;t)\cdot \boldsymbol{n} = E(Q_i(t); Q_j(t)=0)$, then

$$\frac{d}{dt}E(Q_i(t)^2) = 2\left(\lambda_i + \sum_{j\neq i}^{N}\mu_j p_{ji} - \mu_i\right)E(Q_i(t)) + \left(\lambda_i + \sum_{j\neq i}^{N}\mu_j p_{ji} + \mu_i\right) - \mu_i p_i(0;t)$$

$$(4.2)\qquad - \sum_{j\neq i}^{N}\mu_j p_{ji} p_j(0;t) - 2\sum_{j\neq i}^{N}\mu_j p_{ji} E(Q_i(t); Q_j(t)=0).$$

On the other hand,

$$\frac{d}{dt}E(Q_i(t))^2 = 2E(Q_i(t))\frac{d}{dt}E(Q_i(t))$$

$$(4.3)\qquad = 2\left(\lambda_i + \sum_{j\neq i}^{N}\mu_j p_{ji} - \mu_i\right)E(Q_i(t))$$

$$+ 2\mu_i\Gamma_{i;i}(t) - 2E(Q_i(t))\sum_{j\neq i}^{N}\mu_j p_j(0;t)p_{ji}.$$

Subtracting (4.3) from (4.2) then gives the desired result.

Case 3: $\mathrm{Cov}(Q_i(t), Q_j(t))$. We refer back to Proposition 4.1 for the case $I=\{i,j\}$.

$$\frac{d}{dt}\boldsymbol{p}^{ij}(t) = \boldsymbol{p}^{ij}(t)\boldsymbol{A}_{ij} + \sum_{k\neq i,j}^{N}\mu_k \boldsymbol{p}_k^{ij}(0;t)(p_{ki}(\boldsymbol{I}-\boldsymbol{R}_i) + p_{kj}(\boldsymbol{I}-\boldsymbol{R}_j))$$

where

$$\boldsymbol{A}_{ij} = \left(\lambda_i + \sum_{k\neq i,j}^{N}\mu_k p_{ki}\right)\boldsymbol{R}_i + \left(\lambda_j + \sum_{k\neq i,j}^{N}\mu_k p_{kj}\right)\boldsymbol{R}_j$$

$$+ \mu_i\left(q_i + \sum_{k\neq i,j}^{N}p_{ik}\right)\boldsymbol{L}_i + \mu_j\left(q_j + \sum_{k\neq i,j}^{N}p_{jk}\right)\boldsymbol{L}_j$$

$$+ \mu_i p_{ij}\boldsymbol{L}_i\boldsymbol{R}_j - \mu_i\boldsymbol{L}_i\boldsymbol{R}_i + \mu_j p_{ji}\boldsymbol{L}_j\boldsymbol{R}_i - \mu_j\boldsymbol{L}_j\boldsymbol{R}_j$$

$$- (\lambda_i + \lambda_j)\boldsymbol{I} - \sum_{k\neq i,j}^{N}\mu_k(p_{ki} + p_{kj})\boldsymbol{I}$$

$$= \left(\lambda_i + \sum_{k\neq i,j}^{N}\mu_k p_{ki}\right)(\boldsymbol{R}_i - \boldsymbol{I}) + \left(\lambda_j + \sum_{k\neq i,j}^{N}\mu_k p_{kj}\right)(\boldsymbol{R}_j - \boldsymbol{I})$$

$$+ \mu_i\left(q_i + \sum_{k\neq i,j}^{N}p_{ik}\right)\boldsymbol{L}_i + \mu_j\left(q_j + \sum_{k\neq i,j}^{N}p_{jk}\right)\boldsymbol{L}_j$$

$$+ \mu_i p_{ij}\boldsymbol{L}_i\boldsymbol{R}_j - \mu_i\boldsymbol{L}_i\boldsymbol{R}_i + \mu_j p_{ji}\boldsymbol{L}_j\boldsymbol{R}_i - \mu_j\boldsymbol{L}_j\boldsymbol{R}_j.$$

Now $\boldsymbol{R}\boldsymbol{n} = \boldsymbol{n} + \boldsymbol{1}$ and $\boldsymbol{L}\boldsymbol{n} = \boldsymbol{n} - \boldsymbol{1} + \boldsymbol{e}_0$. So consider the quantity $\boldsymbol{n}\otimes\boldsymbol{n}$. If we let the first $\boldsymbol{n}$ be in the ith entry, and the second one in the jth entry, then we get the following set of formulas:

$$\begin{aligned} \boldsymbol{L}_i\boldsymbol{R}_j(\boldsymbol{n}\otimes\boldsymbol{n}) &= (\boldsymbol{n}-\boldsymbol{1}+\boldsymbol{e}_0)\otimes(\boldsymbol{n}+\boldsymbol{1}) \\ &= \boldsymbol{n}\otimes\boldsymbol{n}-\boldsymbol{1}\otimes\boldsymbol{n}+\boldsymbol{e}_0\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\boldsymbol{1}-\boldsymbol{1}\otimes\boldsymbol{1}+\boldsymbol{e}_0\otimes\boldsymbol{1} \\ \boldsymbol{L}_i\boldsymbol{R}_i(\boldsymbol{n}\otimes\boldsymbol{n}) &= \boldsymbol{n}\otimes\boldsymbol{n} \\ (\boldsymbol{R}_i-\boldsymbol{I})(\boldsymbol{n}\otimes\boldsymbol{n}) &= \boldsymbol{1}\otimes\boldsymbol{n}. \end{aligned}$$

There is also a dual set of formulas when the roles of i and j are reversed. Applying $\boldsymbol{n}\otimes\boldsymbol{n}$ to $\boldsymbol{A}_{ij}$, we get

$$\begin{aligned} \boldsymbol{A}_{ij}(\boldsymbol{n}\otimes\boldsymbol{n}) = {} & \left(\lambda_i+\sum_{k\neq i,j}^{N}\mu_k p_{ki}\right)\boldsymbol{1}\otimes\boldsymbol{n}+\left(\lambda_j+\sum_{k\neq i,j}^{N}\mu_k p_{kj}\right)\boldsymbol{n}\otimes\boldsymbol{1} \\ & +\mu_i\left(q_i+\sum_{k\neq i,j}^{N}p_{ik}\right)(\boldsymbol{n}\otimes\boldsymbol{n}-\boldsymbol{1}\otimes\boldsymbol{n}+\boldsymbol{e}_0\otimes\boldsymbol{n}) \\ & +\mu_j\left(q_j+\sum_{k\neq i,j}^{N}p_{jk}\right)(\boldsymbol{n}\otimes\boldsymbol{n}-\boldsymbol{n}\otimes\boldsymbol{1}+\boldsymbol{n}\otimes\boldsymbol{e}_0) \\ & +\mu_i p_{ij}[\boldsymbol{n}\otimes\boldsymbol{n}-\boldsymbol{1}\otimes\boldsymbol{n}+\boldsymbol{e}_0\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\boldsymbol{1}-\boldsymbol{1}\otimes\boldsymbol{1}+\boldsymbol{e}_0\otimes\boldsymbol{1}] \\ & +\mu_j p_{ji}[\boldsymbol{n}\otimes\boldsymbol{n}-\boldsymbol{n}\otimes\boldsymbol{1}+\boldsymbol{n}\otimes\boldsymbol{e}_0+\boldsymbol{1}\otimes\boldsymbol{n}-\boldsymbol{1}\otimes\boldsymbol{1}+\boldsymbol{1}\otimes\boldsymbol{e}_0] \\ & -\mu_i\boldsymbol{n}\otimes\boldsymbol{n}-\mu_j\boldsymbol{n}\otimes\boldsymbol{n} \\ = {} & \left(\lambda_i+\sum_{k\neq i}^{N}\mu_k p_{ki}-\mu_i\right)\boldsymbol{1}\otimes\boldsymbol{n}+\left(\lambda_j+\sum_{k\neq j}^{N}\mu_k p_{kj}-\mu_j\right)\boldsymbol{n}\otimes\boldsymbol{1} \\ & -(\mu_i p_{ij}+\mu_j p_{ji})\boldsymbol{1}\otimes\boldsymbol{1}+\mu_i p_{ij}\boldsymbol{e}_0\otimes\boldsymbol{1}+\mu_j p_{ji}\boldsymbol{1}\otimes\boldsymbol{e}_0 \\ & +\mu_i\boldsymbol{e}_0\otimes\boldsymbol{n}+\mu_j\boldsymbol{n}\otimes\boldsymbol{e}_0. \end{aligned}$$

Consequently,

$$\begin{aligned} \boldsymbol{p}^{ij}(t)\boldsymbol{A}_{ij}(\boldsymbol{n}\otimes\boldsymbol{n}) = {} & \left(\lambda_i+\sum_{k\neq i}^{N}\mu_k p_{ki}-\mu_i\right)E(Q_j(t))+\left(\lambda_j+\sum_{k\neq j}^{N}\mu_k p_{kj}-\mu_j\right)E(Q_i(t)) \\ & +\mu_i E(Q_j(t);Q_i(t)=0)+\mu_j E(Q_i(t);Q_j(t)=0) \\ & -(\mu_i p_{ij}+\mu_j p_{ji})+\mu_i p_{ij}p_i(0;t)+\mu_j p_{ji}p_j(0;t). \end{aligned} \tag{4.4}$$

Furthermore,

$$\begin{aligned} & \sum_{k\neq i,j}^{N}\mu_k \boldsymbol{p}_k^{ij}(0;t)(p_{ki}(\boldsymbol{I}-\boldsymbol{R}_i)+p_{kj}(\boldsymbol{I}-\boldsymbol{R}_j))\boldsymbol{n}\otimes\boldsymbol{n} \\ & \quad = -\sum_{k\neq i,j}^{N}\mu_k \boldsymbol{p}_k^{ij}(0;t)(p_{ki}\boldsymbol{1}\otimes\boldsymbol{n}+p_{kj}\boldsymbol{n}\otimes\boldsymbol{1}) \\ & \quad = -\sum_{k\neq i,j}^{N}\mu_k(p_{ki}E(Q_j(t);Q_k(t)=0)+p_{kj}E(Q_i(t);Q_k(t)=0)). \end{aligned} \tag{4.5}$$

Now we add (4.4) and (4.5) together and subtract off the next two expressions:

$$E(Q_j(t))\frac{d}{dt}E(Q_i(t)) = \Big(\lambda_i + \sum_{k\neq i}\mu_k p_{ki} - \mu_i\Big)E(Q_j(t)) + \mu_i E(Q_j(t))p_i(0;t) - E(Q_j(t))\sum_{k\neq i}^{N}\mu_k p_k(0;t)p_{ki}$$

$$E(Q_i(t))\frac{d}{dt}E(Q_j(t)) = \Big(\lambda_j + \sum_{k\neq j}^{N}\mu_k p_{kj} - \mu_j\Big)E(Q_i(t)) + \mu_j E(Q_i(t))p_j(0;t) - E(Q_i(t))\sum_{k\neq j}^{N}\mu_k p_k(0;t)p_{kj}.$$

After combining these four quantities, we are done.

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