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CALCULATING EXIT TIMES FOR SERIES JACKSON NETWORKS

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Abstract

We define a new family of special functions that we call lattice Bessel functions. They are indexed by the N-dimensional integer lattice such that they reduce to modified Bessel functions when $N = 1$, and the exponential function when $N = 0$. The transition probabilities for an $M/M/1$ queue going from one state to another before becoming idle (exiting at 0) can be solved in terms of modified Bessel functions. In this paper, we use lattice Bessel functions to solve the analogous problem involving the exit time from the interior of the positive orthant of the N-dimensional lattice for a series Jackson network with N nodes. These special functions allow us to derive asymptotic expansions for the taboo transition probabilities, as well as for the tail of the exit-time distribution.

ASYMPTOTIC EXPANSIONS; GROUP SYMMETRY; BESSEL FUNCTIONS; MULTIDIMENSIONAL RANDOM WALKS; TRANSIENT BEHAVIOUR

1. Introduction

Consider the queue length process for the $M/M/1$ system with Poisson arrival rate λ and exponential service rate μ. We can think of it as the reflecting version of the following Markov process. Let $Z(t) = N_\lambda(t) - N_\mu(t) + n$, where $N_\lambda(t)$ and $N_\mu(t)$ are independent Poisson processes with intensities λ and μ respectively. With $N_\lambda(0) = N_\mu(0) = 0$, we have $Z(0) = n$ where n belongs to the set of integers Z. This process is sometimes called a randomized random walk on Z (see Feller [1], p. 59).

If n belongs to Z_+, the set of non-negative integers, then $Z(t)$ behaves the same as the $M/M/1$ queue length process until it hits the zero state. From here, $Z(t)$ may take on negative values, so its state space will be all of Z. Given this relationship, we can think of the busy period for an $M/M/1$ queue as being the time for an identically initialized $Z(t)$ process to be absorbed at the zero state. Let $p_t(m, n)$ be the transition probability for $Z(t)$ starting at m and terminating at n, for time $t > 0$ and all m and n in Z. For m and n in Z_+, we let $q_t(m, n)$ be a similar transition probability from m to n with the additional requirement that

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$Z(s)$ does not equal 0 for $0 \leqq s \leqq t$. It is well known (see Ledermann and Reuter [2]) that $p_t(m, n)$ and $q_t(m, n)$ can be solved explicitly in terms of modified Bessel functions as

$$p_t(m,n) = \exp(-(\lambda+\mu)t)\left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}(n-m)} I_{n-m}(2t\sqrt{\lambda\mu})$$

$$q_t(m,n) = \exp(-(\lambda+\mu)t)\left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}(n-m)} [I_{n-m}(2t\sqrt{\lambda\mu}) - I_{n+m}(2t\sqrt{\lambda\mu})].$$

In this paper, we shall generalize these two results. Just as these formulas describe processes associated with the $M/M/1$ queue length process, we shall construct analogous processes for the N-node series (or pipeline) Jackson network. In particular, the analogue to $q_t(m, n)$ will be the transition probabilities for such a network before *any* one of the servers becomes idle.

To generalize the solutions to $p_t(m, n)$ and $q_t(m, n)$, we need to define a new class of functions that generalize the notion of a modified Bessel function. Recall the following formula for modified Bessel functions:

$$\exp\left(\frac{y}{2}\left[x + \frac{1}{x}\right]\right) = \sum_{n \in Z} x^n I_n(y).$$

For $\boldsymbol{n} = (n_1, \cdots, n_N) \in Z^N$, we say that $I(\boldsymbol{n}, y)$ is a *lattice-Bessel function of rank* N if it is defined by the following generating function relation:

$$\exp\left(\frac{y}{N+1}\left[x_1 + \frac{x_2}{x_1} + \cdots + \frac{x_N}{x_{N-1}} + \frac{1}{x_N}\right]\right) = \sum_{\boldsymbol{n} \in Z^N} x_1^{n_1} \cdots x_N^{n_N} I(\boldsymbol{n}, y).$$

From this defining relation follow various properties of lattice-Bessel functions that we will use in Section 2 and prove in Section 3:

Representations for $I(\boldsymbol{n}, y)$.

$$(1.1) \quad \frac{1}{(2\pi)^N}\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \exp\left[\frac{y}{N+1}(\exp(i\theta_1) + \exp(i(\theta_2 - \theta_1)) + \cdots + \exp(i(\theta_N - \theta_{N-1})) + \exp(-i\theta_N)) - i\sum_{j=1}^{N} n_j\theta_j\right] d\theta_1 \cdots d\theta_N$$

$$(1.2) \quad \sum_{j=0}^{\infty} \prod_{k=0}^{N} \frac{\left(\frac{y}{N+1}\right)^{j + \sum_{l=k+1}^{N} n_l}}{\left(j + \sum_{l=k+1}^{N} n_l\right)!}.$$

Symmetry group. Let G_N be the set of matrices that permute the set of vectors $\{\boldsymbol{e}_1, \boldsymbol{e}_2 - \boldsymbol{e}_1, \cdots, \boldsymbol{e}_N - \boldsymbol{e}_{N-1}, -\boldsymbol{e}_N\}$, where $\boldsymbol{e}_k$ is the kth unit basis vector of Z^N. We then have G_N isomorphic to S_{N+1}, the group of permutations on $N+1$

objects. For all $\boldsymbol{\pi}$ in $\mathrm{Aut}(Z^N)$ we have $I(\boldsymbol{n}, \cdot) = I(\boldsymbol{\pi}(\boldsymbol{n}), \cdot)$ for all $\boldsymbol{n}$ in Z^N, if and only if $\boldsymbol{\pi}$ belongs to G_N. Consequently, G_N is the symmetry group for the rank N lattice-Bessel functions.

Asymptotics. If $y \to \infty^+$, then

$$I(\boldsymbol{n}, y) = \frac{e^y}{\sqrt{N+1}} \left(\frac{N+1}{2\pi y}\right)^{N/2} \left[1 + \left(\frac{N(N+2)}{24} - \frac{1}{2}\boldsymbol{B}(\boldsymbol{n}^*)\right)\frac{1}{y} + O\left(\frac{1}{y^2}\right)\right]$$

where $\boldsymbol{n}^* = (n_1^*, \cdots, n_N^*)$ is derived from $\boldsymbol{n}$ by setting $n_j^* = \sum_{k=j}^N n_k$, and $\boldsymbol{B}(\boldsymbol{n}^*)$ equals the quadratic form $\sum_{j,k=1}^N b_{jk} n_j^* n_k^*$, with $b_{jj} = N$ for all j, and $b_{jk} = -1$ for $j \neq k$.

Consider an N-node series Jackson network with Poisson input rate λ and service rate μ_i for the ith exponential server. Our analogue to $Z(t)$ will be such that the above process is a reflecting version of it. If $N_\lambda(t), N_{\mu_1}(t), \cdots, N_{\mu_N}(t)$ are $N+1$ independent Poisson processes, and $\boldsymbol{Z}(0) = (n_1, \cdots, n_N)$, we define $\boldsymbol{Z}(t)$ to be

$$(N_\lambda(t) - N_{\mu_1}(t) + n_1, N_{\mu_1}(t) - N_{\mu_2}(t) + n_2, \cdots, N_{\mu_{N-1}}(t) - N_{\mu_N}(t) + n_N).$$

Let α and γ be respectively the arithmetic and geometric means of $\lambda, \mu_1, \cdots,$ and μ_N, or

$$\alpha = \frac{\lambda + \sum_{j=1}^N \mu_j}{N+1}, \quad \gamma = \left(\lambda \cdot \prod_{j=1}^N \mu_j\right)^{1/(N+1)}.$$

Now define $\beta_1, \cdots, \beta_N$ so that

$$\lambda = \beta_1 \gamma, \quad \mu_1 = \frac{\beta_2}{\beta_1}\gamma, \cdots, \mu_{N-1} = \frac{\beta_N}{\beta_{N-1}}\gamma, \quad \mu_N = \frac{1}{\beta_N}\gamma.$$

We then get $\beta_1 = \lambda/\gamma$ and $\beta_j = \lambda\mu_1 \cdots \mu_{j-1}/\gamma^j$ for $j = 2, \cdots, N$. For $\boldsymbol{n} \in Z^N$, let $\boldsymbol{\beta}^{\boldsymbol{n}} = \prod_{i=1}^N \beta_i^{n_i}$. We can now write the transient behavior for $\boldsymbol{Z}(t)$ compactly.

Theorem 1.1. *If* $p_t(\boldsymbol{m}, \boldsymbol{n}) = \Pr\{\boldsymbol{Z}(t) = \boldsymbol{n} \mid \boldsymbol{Z}(0) = \boldsymbol{m}\}$ *for all* $\boldsymbol{m}$ *and* $\boldsymbol{n}$ *in* Z^N, *then*

$$p_t(\boldsymbol{m}, \boldsymbol{n}) = \exp(-(N+1)\alpha t)\boldsymbol{\beta}^{\boldsymbol{n}-\boldsymbol{m}} \cdot I(\boldsymbol{n} - \boldsymbol{m}, (N+1)\gamma t).$$

If $\boldsymbol{Z}(0)$ equals an element in the interior of Z_+^N, where each component is strictly positive, then let T equal the first time that $\boldsymbol{Z}(t)$ has a zero component. We then call T a random stopping time. The joint distribution of $\boldsymbol{Z}(t)$ with the event $\{T > t\}$ represents a new Markov process which acts like $\boldsymbol{Z}(t)$ but is absorbed when it touches the boundary of Z_+^N. In queueing theoretic terms, T equals the time until one of the queues becomes empty in a Jackson series network.

Theorem 1.2. *If* $q_t(\boldsymbol{m}, \boldsymbol{n}) = \Pr\{Z(t) = \boldsymbol{n},\ T > t \mid Z(0) = \boldsymbol{m}\}$ *for all* $\boldsymbol{m}$ *and* $\boldsymbol{n}$ *in* Z_+^N, *then*

$$q_t(\boldsymbol{m}, \boldsymbol{n}) = \exp(-(N+1)\alpha t)\boldsymbol{\beta}^{\boldsymbol{n}-\boldsymbol{m}} \sum_{\pi \in G_N} (-1)^\pi I(\boldsymbol{n} - \boldsymbol{\pi}(\boldsymbol{m}), (N+1)\gamma t), \tag{1.3}$$

where $(-1)^\pi$ *is the sign of* $\boldsymbol{\pi}$, *viewed as a permutation.*

Given $p_t(\boldsymbol{m}, \boldsymbol{n})$ and $q_t(\boldsymbol{m}, \boldsymbol{n})$ in terms of lattice-Bessel functions, we can now derive asymptotic formulas for them.

Theorem 1.3. *For* $t \uparrow \infty$, *we have*

$$p_t(\boldsymbol{m}, \boldsymbol{n}) = \boldsymbol{\beta}^{\boldsymbol{n}-\boldsymbol{m}} \frac{\exp(-(N+1)(\alpha-\gamma)t)}{\sqrt{N+1}(2\pi\gamma t)^{N/2}} \times \left[1 + \left(\frac{N(N+2)}{24(N+1)\gamma} - \frac{B(\boldsymbol{n}^* - \boldsymbol{m}^*)}{2(N+1)\gamma}\right)\frac{1}{t} + O\left(\frac{1}{t^2}\right)\right]$$

$$q_t(\boldsymbol{m}, \boldsymbol{n}) = -\boldsymbol{\beta}^{\boldsymbol{n}-\boldsymbol{m}} \frac{\exp(-(N+1)(\alpha-\gamma)t)}{2(N+1)^{3/2}\gamma t(2\pi\gamma t)^{N/2}} \times \left[\sum_{\pi \in G_N} (-1)^\pi \boldsymbol{B}(\boldsymbol{n}^* - \boldsymbol{\pi}(\boldsymbol{m})^*) + O\left(\frac{1}{t}\right)\right].$$

Moreover, if $\beta_i < 1$ *for all* i, *then*

$$\Pr\{T > t \mid Z(0) = \boldsymbol{m}\} = O\left(\frac{\exp(-(N+1)(\alpha-\gamma)t)}{t\sqrt{t^N}}\right).$$

2. Calculating transition probabilities

Let $l_1(Z)$ be the Banach space of doubly infinite sequences that are absolutely summable. We define $\boldsymbol{e}_n$ for $n \in Z$ to be the nth unit basis vector for $l_1(Z)$. Every sequence $\{\cdots, a_{-1}, a_0, a_1, \cdots\}$ can then be written as $\Sigma_{n \in Z} a_n \boldsymbol{e}_n$. On $l_1(Z)$, we define right and left shift operators which we denote as $\boldsymbol{R}$ and $\boldsymbol{L}$ respectively. They can be defined in terms of the $\boldsymbol{e}_n$'s where $\boldsymbol{e}_n\boldsymbol{R} = \boldsymbol{e}_{n+1}$ and $\boldsymbol{e}_n\boldsymbol{L} = \boldsymbol{e}_{n-1}$. It then follows that $\boldsymbol{RL} = \boldsymbol{LR} = \boldsymbol{I}$ so $\boldsymbol{R}^{-1} = \boldsymbol{L}$.

The joint distribution for $\boldsymbol{Z}(t)$ can be encoded as a vector belonging to $l_1(Z)^{(N)}$, then N-fold tensor product of $l_1(Z)$ with itself. A basis for this space is the set of $\bigotimes_{i=1}^N \boldsymbol{e}_{n_i}$'s where $\boldsymbol{n} = (n_1, \cdots, n_N)$ ranges over all Z^N. Given $\boldsymbol{m}$ in Z^N, we define $\boldsymbol{p}_{\boldsymbol{m}}(t)$ to be an $l_1(Z)^{(N)}$-vector that represents the distribution of $\boldsymbol{Z}(t)$ given that $\boldsymbol{Z}(0) = \boldsymbol{m}$, or

$$\boldsymbol{p}_{\boldsymbol{m}}(t) = \sum_{\boldsymbol{n} \in Z^N} p_t(\boldsymbol{m}, \boldsymbol{n}) \boldsymbol{e}_{n_1} \otimes \cdots \otimes \boldsymbol{e}_{n_N}.$$

We let $\boldsymbol{P}(t)$ be the operator that maps $\bigotimes_{i=1}^N \boldsymbol{e}_{m_i}$ into $\boldsymbol{p}_{\boldsymbol{m}}(t)$. Thus $\boldsymbol{P}(t)$ encodes all of the transition probabilities for $Z(t)$. Let R_i and L_i be operators acting on $l_1(Z)^{(N)}$ where $R_i = I \otimes \cdots \otimes R \otimes \cdots \otimes I$ (R occurring only in the ith position) and L_i is defined similarly. The forward and backward equations for $Z(t)$ can then be written compactly as

$$\frac{d}{dt} P(t) = P(t)A = AP(t)$$

with $P(0) = I$, and

$$A = \lambda R_1 + \mu_1 L_1 R_2 + \cdots + \mu_{N-1} L_{N-1} R_N + \mu_N L_N - \left(\lambda + \sum_{i=1}^{N} \mu_i\right) I.$$

For more details on this tensor representation, see [3].

Proof of Theorem 1.1. Given the definition of α, γ, and the β_i's we have

$$A = \gamma\left(\beta_1 R_1 + \frac{\beta_2}{\beta_1} L_1 R_2 + \cdots + \frac{1}{\beta_N} L_N\right) - (N+1)\alpha I.$$

By the forward (or backward) equations we have $P(t) = \exp(tA)$, and $L_i = R_i^{-1}$, so using the generating function relation gives us

$$\begin{aligned} P(t) &= \exp\left(t\left[\gamma\left(\beta_1 R_1 + \cdots + \frac{1}{\beta_N} L_N\right) - (N+1)\alpha I\right]\right) \\ &= \exp(-(N+1)\alpha t)\exp\left(\frac{(N+1)\gamma t}{N+1}\left(\beta_1 R_1 + \cdots + \frac{1}{\beta_N} L_N\right)\right) \\ &= \sum_{n \in Z^N} \exp(-(N+1)\alpha t)\beta^n I(n,(N+1)\gamma t) \cdot \prod_{i=1}^{N} R_i^{n_i} \end{aligned}$$

and the formula for $p_t(m, n)$ follows.

Proof of Theorem 1.2. We shall define $q_t(m, n)$ by (1.3) and then show that it is the desired probability by being the unique solution to the forward and backward equations. Using Theorem 1.2 and the symmetry group for lattice-Bessel functions, we have four representations for $q_t(m, n)$:

(2.1) $$q_t(m, n) = \exp(-(N+1)\alpha t)\beta^{n-m} \sum_{\pi \in G_N} (-1)^{\pi} I(n - \pi(m), (N+1)\gamma t)$$

(2.2) $$= \exp(-(N+1)\alpha t)\beta^{n-m} \sum_{\pi \in G_N} (-1)^{\pi} I(\pi(n) - m, (N+1)\gamma t)$$

(2.3) $$= \sum_{\pi \in G_N} (-1)^{\pi} \beta^{\pi(m)-m} p_t(\pi(m), n)$$

(2.4) $$= \sum_{\pi \in G_N} (-1)^{\pi} \beta^{n-\pi(n)} p_t(m, \pi(n)).$$

By (2.1) and (2.2) we have $q_t(m, n) = 0$ whenever m or n belongs to the boundary of Z_+^N. This follows from the fact that any such vector is left invariant by some transposition (an odd order 2 element). If τ permutes only $e_i - e_{i-1}$ and $e_{i+1} - e_i$ (with the obvious modifications for $i = 1$ or N) then $\tau(\boldsymbol{n}) = \boldsymbol{n}$ if $n_i = 0$, since in general

$$\boldsymbol{n} = \left(\sum_{i=1}^{N} n_i\right) e_1 + \left(\sum_{i=2}^{N} n_i\right)(e_2 - e_1) + \cdots + n_N(e_N - e_{N-1}).$$

For every odd permutation π in G_N, we can then choose a unique even one, π' such that $\pi(\boldsymbol{n}) = \pi'(\boldsymbol{n})$. Since $(-1)^{\pi} + (-1)^{\pi'} = 0$, we get $q_t(\boldsymbol{m}, \boldsymbol{n}) = 0$ for such an $\boldsymbol{n}$ (or $\boldsymbol{m}$). By (2.3), it is immediate that $(d/dt)\boldsymbol{Q}_t = \boldsymbol{Q}_t\boldsymbol{A}$ and (2.4) gives us $(d/dt)\boldsymbol{Q}_t = \boldsymbol{A}\boldsymbol{Q}_t$. Therefore, the $q_t(\boldsymbol{m}, \boldsymbol{n})$ solve the same forward and backward equations as $\boldsymbol{P}_t$, where $q_t(\boldsymbol{m}, \boldsymbol{n}) = 0$ whenever $\boldsymbol{n}$ or $\boldsymbol{m}$ is on the boundary of Z_+^N. Hence the $q_t(\boldsymbol{m}, \boldsymbol{n})$ for $\boldsymbol{m}$ and $\boldsymbol{n}$ in Z_+^N solve the same differential difference equations as $\Pr\{\boldsymbol{Z}(t) = \boldsymbol{n},\ T > t \mid \boldsymbol{Z}(0) = \boldsymbol{m}\}$. By uniqueness, we have equality.

Proof of Theorem 1.3. The first two statements follow immediately from Theorems 1.1 and 1.2 by using the asymptotic properties of lattice-Bessel functions. For the last statement, we have

$$\begin{aligned}\Pr\{T > t \mid \boldsymbol{Z}(0) = \boldsymbol{m}\} &= \sum_{\boldsymbol{n} \in Z_+^N} q_t(\boldsymbol{m}, \boldsymbol{n}) \\ &= \sum_{\boldsymbol{n} \in Z_+^N} \boldsymbol{\beta}^{\boldsymbol{n}} \cdot O\left(\frac{\exp(-(N+1)(\alpha - \gamma)t)}{t\sqrt{t^N}}\right)\end{aligned}$$

and $\sum_{\boldsymbol{n} \in Z_+^N} \boldsymbol{\beta}^{\boldsymbol{n}} = \prod_{i=1}^{N} \sum_{n_i=0}^{\infty} \beta_i^{n_i} < \infty$ if and only if $\beta_i < 1$ for all i.

We conclude by noting that for $N = 1$, $(N+1)(\alpha - \gamma) = \lambda + \mu - 2\sqrt{\lambda\mu}$ which is the relaxation parameter for the $M/M/1$ queue.

3. Properties of lattice-Bessel functions

Proposition 3.1. *We have the following representations for* $I(\boldsymbol{n}, y)$:

$$I(\boldsymbol{n}, y) = \frac{1}{(2\pi)^N} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \exp\left[\frac{y}{N+1}(\exp(i\theta_1) + \exp(i(\theta_2 - \theta_1)) + \cdots + \exp(-i\theta_N)) - i\sum_{j=1}^{N} n_j\theta_j\right] d\theta_1 \cdots d\theta_N$$

$$I(\boldsymbol{n}, y) = \sum_{j=0}^{\infty} \prod_{k=0}^{N} \frac{\left(\frac{y}{N+1}\right)^{j + \sum_{l=k+1}^{N} n_l}}{\left(j + \sum_{l=k+1}^{N} n_l\right)!}.$$

Proof. The integral representation follows immediately from employing the generating function definition of $I(\boldsymbol{n}, y)$, setting $x_j = \exp(i\theta_j)$, and using the orthogonality properties of the functions $\{\exp(in_j\theta_j)\}_{n_j \in Z}$ when integrated over $[-\pi, \pi]$.

For deriving the power series representation, we adopt the convention that $1/n! = 0$ when n is negative. This will allow us not to be concerned with the range of summation in certain cases. For example, by this convention we have $\sum_{n=0}^{\infty} 1/n! = \sum_{n \in Z} 1/n!$.

$$\begin{aligned}
\exp\left(\frac{y}{N+1}\left(x_1 + \cdots + \frac{1}{x_N}\right)\right) &= \sum_{j=0}^{\infty}\left(\frac{y}{N+1}\right)^j \frac{1}{j!}\left(x_1 + \cdots + \frac{1}{x_N}\right)^j \\
&= \sum_{j=0}^{\infty}\left(\frac{y}{N+1}\right)^j \sum_{m_1+\cdots+m_{N+1}=j} \frac{x_1^{m_1-m_2}\cdots x_N^{m_N-m_{N+1}}}{m_1!\cdots m_{N+1}!} \\
&= \sum_{(m_1,\cdots,m_N)\in Z^N} \sum_{j_*=0}^{\infty}\left(\frac{y}{N+1}\right)^{j_*+\sum_{k=1}^N m_k} \frac{x_1^{m_1-m_2}\cdots x_N^{m_N-j_*}}{m_1!\cdots m_N!\, j_*!} \\
&= \sum_{(n_1,\cdots,n_N)\in Z^N} x_1^{n_1}\cdots x_N^{n_N} \sum_{j_*=0}^{\infty} \prod_{k=1}^{N+1} \frac{\left(\frac{y}{N+1}\right)^{j_*+\sum_{l=k}^N n_l}}{\left(j_* + \sum_{l=k}^N n_l\right)!}.
\end{aligned}$$

For the last step, notice that if $n_1 = m_1 - m_2, n_2 = m_2 - m_3, \cdots,$ and $n_N = m_N - j_*$, then for all $j, m_j = j_* + \sum_{k=j}^N n_k$.

Proposition 3.2. *Define* $G_N = \{\pi \mid \pi \in \mathrm{Aut}(Z^N)$ *and* $I(\pi(n), \cdot) = I(n, \cdot)$ *for all* $n \in Z^N\}$ *then* $G_N \cong S_{N+1}$, *the group of permutations on* $N+1$ *objects.*

Proof. Let e_i be the ith unit vector in Z^N. We prove this result by showing that each element in G_N is uniquely defined by a permutation on the '$N+1$ objects' $\{e_1, e_2 - e_1, \cdots, e_N - e_{N-1}, -e_N\}$.

If $x^n = \prod_{i=1}^N x_i^{n_i}$, then

$$f(x) = x^{e_1} + x^{e_2-e_1} + \cdots + x^{e_N-e_{N-1}} + x^{-e_N} = x_1 + \frac{x_2}{x_1} + \cdots + \frac{x_N}{x_{N-1}} + \frac{1}{x_N}. \tag{3.1}$$

Let $\tilde{x}_j = x^{\pi(e_j)}$ for $j = 1, \cdots, N$ then $\tilde{x}^n = x^{\pi(n)}$. We then have $f(\tilde{x}) = f(x)$ if and only if π permutes the set $\{e_1, e_2 - e_1, \cdots, -e_N\}$. Moreover,

$$\begin{aligned}
\exp\left(\frac{y}{N+1} f(x)\right) &= \exp\left(\frac{y}{N+1} f(\tilde{x})\right) \\
&= \sum_{n \in Z^N} \tilde{x}^n I(n, y) \\
&= \sum_{n \in Z^N} x^{\pi(n)} I(n, y) \\
&= \sum_{n \in Z^N} x^n I(\pi^{-1}(n), y)
\end{aligned}$$

and so $I(\boldsymbol{n}, \cdot) = I(\boldsymbol{\pi}^{-1}(\boldsymbol{n}), \cdot)$ for all $\boldsymbol{n}$ in Z^N and all $\boldsymbol{\pi}^{-1}$ (or $\boldsymbol{\pi}$) in G_N.

Deriving the asymptotic expansion for $I(\boldsymbol{n}, y)$ as $y \to \infty^+$ would take us too far afield from the main theme of this paper. This will be done in a forthcoming paper (see [4]). For now, we prove a weaker result.

Proposition 3.4.

$$\lim_{y\to\infty^+} \frac{\sqrt{y}^N}{e^y} I(\boldsymbol{n}, y) = \frac{1}{\sqrt{N+1}} \left(\frac{N+1}{2\pi}\right)^{N/2}. \tag{3.2}$$

Proof. Let f be as defined above in (3.1). If $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_N)$, $d\boldsymbol{\theta} = d\theta_1 \cdots d\theta_N$, and $\exp(i\boldsymbol{\theta}) = (e^{i\theta_1}, \cdots, e^{i\theta_N})$, we can rewrite (1.1) as

$$I(\boldsymbol{n}, y) = \frac{1}{(2\pi)^N} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \exp\left(\frac{y}{N+1} f(\exp(i\boldsymbol{\theta})) - i\boldsymbol{n} \cdot \boldsymbol{\theta}\right) d\boldsymbol{\theta}.$$

By (1.2), it is clear that $I(\boldsymbol{n}, y)$ is a real-valued function for real y, so

$$I(\boldsymbol{n}, y) = \frac{1}{(2\pi)^N} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \exp\left(\frac{y}{N+1} \mathrm{Re}(f(\exp(i\boldsymbol{\theta})))\right)$$
$$\times \cos\left(\frac{y}{N+1} \mathrm{Im}(f(\exp(i\boldsymbol{\theta}))) - \boldsymbol{n} \cdot \boldsymbol{\theta}\right) d\boldsymbol{\theta}.$$

For each natural number k, take $\boldsymbol{\theta}^{(k)}$ to represent some homogeneous multinomial of degree k. If we define $g(\boldsymbol{\theta})$ to equal

$$g(\boldsymbol{\theta}) = \tfrac{1}{2}[\theta_1^2 + (\theta_2 - \theta_1)^2 + \cdots + \theta_N^2],$$

we can expand $f(\exp(i\boldsymbol{\theta}))$ as follows:

$$\mathrm{Re}(f(\exp(i\boldsymbol{\theta}))) = N + 1 - g(\boldsymbol{\theta}) + O(\boldsymbol{\theta}^{(4)})$$
$$\mathrm{Im}(f(\exp(i\boldsymbol{\theta}))) = O(\boldsymbol{\theta}^{(3)}).$$

Letting $u_j = \theta_j \sqrt{y}$ for all j, and making the above substitutions, we get

$$I(\boldsymbol{n}, y) = \frac{e^y}{(2\pi\sqrt{y})^N} \int_{-\pi\sqrt{y}}^{\pi\sqrt{y}} \cdots \int_{-\pi\sqrt{y}}^{\pi\sqrt{y}} \exp\left(\frac{-1}{N+1} g(\boldsymbol{u}) + O\left(\frac{\boldsymbol{u}^{(4)}}{y}\right)\right)$$
$$\times \cos\left(O\left(\frac{\boldsymbol{u}^{(3)}}{\sqrt{y}}\right) - \frac{\boldsymbol{n} \cdot \boldsymbol{u}}{\sqrt{y}}\right) d\boldsymbol{u}.$$

From this, if follows that

$$\lim_{y\to\infty^+} \frac{\sqrt{y}^N}{e^y} I(\boldsymbol{n}, y) = \frac{1}{(2\pi)^N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left(\frac{-1}{N+1} g(\boldsymbol{u})\right) d\boldsymbol{u}.$$

Now $g(\boldsymbol{u})$ can be written as the quadratic form $\frac{1}{2}\sum_{j,k=1}^N c_{jk} u_j u_k$, where $\boldsymbol{C} = \{c_{jk} \mid 1 \leqq j,\ k \leqq N\}$ equals

$$c_{jk} = \begin{cases} 2 & j = k \\ -1 & |j-k| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

C is a positive definite, symmetric matrix. This means that C can be diagonalized by a unitary matrix. If D is that diagonal matrix, and $d_1, \cdots, d_N$ are its non-zero (diagonal) entries, we get

$$\begin{aligned} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left(\frac{-1}{N+1} g(\boldsymbol{u})\right) d\boldsymbol{u} &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left(\frac{-1}{2(N+1)} \boldsymbol{u}\boldsymbol{C}\boldsymbol{u}^{\mathrm{T}}\right) d\boldsymbol{u} \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left(\frac{-1}{2(N+1)} \tilde{\boldsymbol{u}}\boldsymbol{D}\tilde{\boldsymbol{u}}^{\mathrm{T}}\right) d\tilde{\boldsymbol{u}} \\ &= \prod_{j=1}^{N} \int_{-\infty}^{\infty} \exp\left(\frac{-d_j}{2(N+1)} \tilde{u}_j^2\right) d\tilde{u}_j \\ &= \prod_{j=1}^{N} \left(\frac{2\pi(N+1)}{d_j}\right)^{1/2} \\ &= \frac{1}{\sqrt{\det(C)}} [2\pi(N+1)]^{N/2}. \end{aligned}$$

Expanding by minors, we can show by induction that $\det(C) = N+1$. Making this substitution and dividing by $(2\pi)^N$ gives us (3.2).

From this proposition, it follows that as $t \to \infty^+$,

$$p_t(\boldsymbol{m}, \boldsymbol{n}) = O\left(\frac{\exp(-(N+1)(\alpha-\gamma)t)}{\sqrt{t^N}}\right)$$

and since the limit in (3.2) is independent of the index $\boldsymbol{n}$, we also have

$$r_t(\boldsymbol{m}, \boldsymbol{n}) = o\left(\frac{\exp(-(N+1)(\alpha-\gamma)t)}{\sqrt{t^N}}\right).$$

These asymptotics clearly suggest the stronger results of Theorem 1.3.

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